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Sixth Term Examination Paper [STEP]

Mathematics 1 [9465]

2018

Examiner's Report

Hints and Solutions

Mark Scheme

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STEP MATHEMATICS 1

2018

Examiner's Report

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General Comments

In order to get the fullest picture, this document should be read in conjunction with the question paper, the marking scheme and (for comments on the underlying purpose and motivation for finding the right solution-approaches to questions) the *Hints and Solutions* document.

The purpose of the STEPs is to learn what students are able to achieve mathematically when applying the knowledge, skills and techniques that they have learned within their standard A-level (or equivalent) courses … but seldom within the usual range of familiar settings. STEP questions require candidates to work at an extended piece of mathematics, often with the minimum of specific guidance, and to make the necessary connections. This requires a very different mind-set to that which is sufficient for success at A-level, and the requisite skills tend only to develop with prolonged and determined practice at such longer questions.

One of the most crucial features of the STEPs is that the routine technical and manipulative skills are almost taken for granted; it is necessary for candidates to produce them with both speed and accuracy so that the maximum amount of time can be spent in thinking their way through the problem and the various hurdles and obstacles that have been set before them. Most STEP questions begin by asking the solver to do something relatively routine or familiar before letting them loose on the real problem. Almost always, such an opening has not been put there to allow one to pick up a few easy marks, but rather to point the solver in the right direction for what follows. Very often, the opening result or technique will need to be used, adapted or extended in the later parts of the question, with the demands increasing the further on that one goes. So a candidate should never think that they are simply required to 'go through the motions'; rather they will, sooner or later, be required to show either genuine skill or real insight in order to make a reasonably complete effort. The more successful candidates are the ones who manage to figure out how to move on from the given starting-point.

Finally, reading through a finished solution is often misleading – even unhelpful – unless you have attempted the problem for yourself. This is because the thinking has been done for you. So, when you read through the report and look at the solutions (either in the mark scheme or the *Hints and Solutions* booklet), try to figure out how you could have arrived at the solution, learn from your mistakes and pick up as many tips as you can whilst working through past paper questions.

This year far too many candidates wasted time by attempting more than six questions, with many of these candidates picking up 0-4 marks on several 'false starts' which petered out the moment some understanding was required.

There were almost 2000 candidates for this SI paper. Almost one-sixth of candidates failed to reach a total of 30 and around two-thirds fell below half-marks overall. This highlights the fact that many candidates don't find this test an easy one. At the other end of the spectrum, almost one-in-ten managed a total of 84 out of 120 – these candidates usually marked out by their ability to complete whole questions – with almost 4% of the entry achieving the highly praiseworthy feat of getting into three-figures with their overall score.

The paper is constructed so that question 1 is very approachable indeed, the intention being to get everyone started with some measure of success; unsurprisingly, Q1 was the most popular question of all, with almost all candidates attempting it, and it also turned out to be the most successful question on the paper with a mean score of more than 15 out of 20. Around 7% of candidates *didn't* make any kind of attempt at it at all.

In order of popularity, Q1 was followed by Qs. 2, 7, 4 and 3. Indeed, it was the pure maths questions in Section A that attracted the majority of attention from candidates, with the most popular applied question (Q9, mechanics) still getting fewer 'hits' than the least popular pure question (Q5). Questions 10, 11 and 13 proved to attract very little attention from candidates and many of the attempts were minimal.

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Question 1

This was the most popular question and elicited the highest average score of any question on the paper. This is hardly surprising, given that it is the question that most closely resembles a long question from an A-level paper and that its demands stand out as, at least, manageable. This proved to be the case with almost all takers, the most noticeable shortcomings arising with the inequality result at the end: many candidates didn't seem to realise that it is (almost) always easier to consider the sign of some variable quantity than to prove that one variable is greater or less than another. In this case, all it takes is to turn $S > \frac{1}{3}T$ into either $S - \frac{1}{3}T > 0$ or $3S - T > 0$.

Another common fault amongst candidates is to arrive at a *given* answer (there are two here) without producing thoroughly convincing working to support it.

The final point to be made here is that a diagram (even a hastily drawn one that covers the basic features required) can be of immense value, even when not directly requested. In this question, rather a lot of solutions arose from (effectively) mistakenly having the point R on the x-axis.

Question 2

This question highlighted that most people had less than an adequate understanding of inequalities. Many students decided to turn the given statements into equalities (such as $\pi = 10$) and work through before finally, optimistically, inserting an inequality at the end. Others reversed inequalities when taking logs. Only a tiny number considered why taking logs of an inequality was acceptable.

Although there were many possible choices for the base of logarithms to work with, blindly manipulating things until the result appears was unlikely to get to the correct answer. Too many candidates produced a "stream of consciousness" including many correct statements but without any sense of creating an argument. Mathematics should be about communicating ideas. In particular, it is clear that many candidates did not know that it is flawed to start at the required result and manipulate until a known theorem is reached. Particularly when working with inequalities, these types of arguments are not always reversible.

Nonetheless, Q2 was the second most popular question and with the second highest mean score (just over half marks); this was mostly as a result of being given so much information upfront.

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Question 3

Most candidates did reasonably well on part (i), getting almost full marks for that section, but there were relatively few attempts at (ii). Surprisingly, many candidates who did attempt part (ii) didn't realise that the best approach was to reverse their reasoning from part (i), and tried a completely different method, in some cases successfully.

In part (i), while the tangent solution was by far the most common successful approach, several other correct trigonometry solutions were found. However, most students who didn't use tangents made little useful progress and represented an overwhelming majority of attempts. A common slip in all these methods was cancelling a factor from both sides of an equation without mentioning that it was non-zero. There were also a very small number of elegant geometrical solutions obtained by adding another line to the diagram (either the angle bisector at S or the reflection of PS in the vertical line through P).

The paucity of attempts at part (ii) may well have been because there was no obvious way to proceed, but also possibly because some candidates confused the two directions of implication, thinking that $\beta = 2\alpha$ followed from part (i). While there was no requirement to sketch the hyperbola $y^2 = 3(x^2 - a^2)$, there were several incorrect sketches (ellipses, modified parabolae, etc.). Candidates who correctly found "$\beta = 2\alpha$ or $\beta = 2\alpha - \pi$" often did not clearly justify how they eliminated other possibilities.

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Question 4

This was a very popular question, yet elicited a mean score of under 6 out of 20. The principal reason for this is that most attempts recognised that one should differentiate to start with; after that, efforts tailed off very rapidly indeed. So, in general, the first part was answered well with candidates showing a good systematic (although not always efficient) approach to differentiation. However, too many candidates thought that $(\ln x)^2 = 1$ was equivalent to $\ln x = 1$.

When integrating, the majority of candidates forgot the modulus sign in $\int \frac{1}{x}\, dx = \ln|x| + c$. It did cause many candidates to come unstuck when dealing with $\ln(-1)$ – with several candidates trying to ignore this term entirely which made the rest of the question trickier than should have been the case.

Very few candidates seemed to realise that nearly all the features of the final graph could be deduced from facts given in the question rather than the outcome of their integration. Many candidates did not link the facts given about f(x) with their sketches.

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Question 5

Around a third of all candidates made an attempt at this question, yet it had the second lowest mean score of around only 4 marks out of 20.

To begin with, many candidates seemed to think that every polynomial begins with a coefficient of 1, and this led to serious difficulties with part (ii). Part (ii) also involved a *proof by contradiction* (or equivalent) and the exercise of choice … and most candidates aren't happy with finding one answer of what might be many. Those who spotted that $r = 2$ worked then frequently failed to verify it clearly.

Part (iii) was all about factors and the biggest obstacle to success lay in the widespread inability to think of integer factors as being anything other than positive; thus, the statement that 17 couldn't have four distinct factors – whilst correct – was almost invariably given because the candidate could only think of 1 and 17. In fact, 17 *does* have four integer factors, but the reason the 'proof' failed was actually because +17 and –17 couldn't both be used.

Most who persevered this far seemed to realise that factors of 16 were to be used but failed to give all the correct possibilities (or to show why there were no others) as a result of a poorly devised system for enumerating them.

Question 6

Half of all candidates attempted the question and the mean score was around 6.7 out of 20. The big issue here was that, while almost all attempts got off to a successful start, sooner or later they began to flounder as the technical demands increased. The opening results and part (i) were generally handled very well but B_n's result in (ii) required a bit more effort to finish it off, since the use of the initial result had to be supplemented with the trivial, but somewhat disguised, identity that $\cos(\pi - \theta) = -\cos\theta$ before the given answer could be obtained.

Thereafter, solutions generally fell into one of two groups: those who used trig. identities to complete the question and those who didn't. Those in the former group were much more successful than their counterparts in the latter group, most of whom made little progress and gave up.

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Question 7

As with most of Qs. 3-8, this was a very popular question, but with a low mean score; in this case, of around 6 out of 20. Part (i) was generally done well although quite a lot of candidates made an algebraic error here. It was then possible to obtain full marks in the remaining sections by follow through, but few candidates followed through correctly. There were also a few candidates who found the z^3 and constant terms, but didn't actually check that the other terms cancelled. A rare alternative method to (i) was to write z in terms of x and substitute into $az^3 + b = 0$; where this method was used, it was generally well-executed.

A very large majority of candidates attempting part (ii) did so "backwards", showing instead that $d^2 > 4c^3$ follows from the relations between c, d and p, q. Those who correctly interpreted this part of the question often failed to justify strictness of the inequality.

There were more essentially correct responses to part (iii) as candidates were able to identify appropriate values of p and q, although arithmetic errors were common.

Few candidates made progress with part (iv), but those who did often got 3 marks out of 4 because they did not write the solutions in terms of c and/or d.

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Question 8

Only two of the eight pure questions drew attempts from under 1000 candidates and this was one of them. Its mean score of 7.3 out of 20 made it the third most successful Section A question after Qs.1 & 2. Part of its success lay in the fact that – for those who were able to get past the sense that this had something to do with sine and cosine – it was actually a straightforward test of calculus in some 'pure' way, unadulterated by actual (known) functions.

Naturally, the differentiations in the early parts were more competently handled than the integrations of the latter parts, but there were many candidates who manipulated integrations by 'parts' and 'substitution' with considerable skill. Making all the connections, as results accumulated, wasn't too easy but those who persevered to the end scored very well indeed. However, those with "uncertainties" in their understanding of the various calculus approaches made little progress.

Question 9

This was the most popular of the applied questions, drawing more than 700 attempts. It was, after Qs.1 & 2, the highest-scoring question, one of only the three to average more than 10 out of 20.

Most candidates realised that an energy argument was useful in the first part of the question, but there were some spurious reasons given for the direction of the inequality. Candidates were then generally able to apply kinematics formulae to make progress; however, reaching the required form proved challenging. In particular, candidates were often unclear about the sign conventions they were using. When taking square roots in mechanics – either directly or in the context of the quadratic equation – the plus or minus solutions often have two physical interpretations and candidates were very bad at explaining why they were selecting one of the signs.

The algebra required to get to the given form was beyond most candidates. However, they could still have used the given form to answer the last part using some basic calculus.

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Question 10

This question was neither popular nor well-handled by those starting it; part of the problem was that there appears to be too many things to consider … two engines and an unspecified number of carriages. A mean score of only just over 4 out of 20 is actually rather misleading here, since most of those attempts consisted of little more than an unused (and usually incomplete) diagram. Of those who made a serious attempt, most started this question well, considering Newton's 2nd Law for the front engine and the whole train. However, in later parts very few candidates considered a general carriage between the two engines which made any argument quite difficult.

Candidates generally had difficulty in linking the condition that $k > \frac{1}{2}n$ in any meaningful way because they were not sufficiently systematic in their consideration of carriages between the two engines and carriages after the two engines. Some candidates effectively did part (iii) before part (ii) and this was perfectly acceptable.

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Question 11

A good diagram was a very important starting point for this question, but too many candidates did not use one or it was too small to be of any use. Interpreting the question using a diagram is a vital skill which we would advise requires practice. Candidates also often fooled themselves by drawing a misleading diagram – for example, assuming that BOA was a right angle.

In (ii) lots of candidates noticed that setting $\theta = 2\alpha$ gave the required result, however this selection was rarely justified in the context of the question. Equally the physical significance of the reaction force being zero was not clearly communicated by candidates.

In part (iii) candidates who resolved forces and used some trigonometric identities generally scored well, although justification for the condition being necessary was very rare.

The final part was beyond most candidates. This was the least popular question on the paper and the one with the lowest mean score.

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Question 12

This was a reasonably popular question, with almost a third of the candidature making an attempt at it. Most of these candidates did (i) and (ii) well. It was quite common to state $N_1 \sim \text{Bin}(2, p)$ and write down the mean and variance of that. This was acceptable but, because the variance is given in the question, we required candidates to explicitly state the distribution rather than just giving the answers.

A lot of candidates struggled with (iii), commonly thinking it was still binomial or failing to condition on the chosen coins. While the most common successful approach was to calculate the combined probabilities of different numbers of heads, some candidates found the mean correctly by averaging the means of pairs of coins. It was very rare to obtain the variance correctly with this latter approach.

There were relatively few attempts at part (iv) which got the correct expression for the difference of the variances, and very few came up with the idea of completing the square. There was a very small number – no more than 5 – of correct solutions by the AM-GM inequality or more exotic methods. The condition for equality was often not justified.

Question 13

Part (i) was well done overall, though some candidates just listed (many) cases, often still getting the correct probabilities, and some compared expected scores (which received no credit). It was very common to assume $n = 2$ for part or all of the question and little credit was available for this (at most 5 out of 11 for part (i), 1 out of 4 for part (ii), and 4 out of 5 for part (iii)).

Many candidates did part (ii) well (marks were available here for following through wrong probabilities from (i)), but most did not consider a cancelling factor of $\frac{1}{6}$ arising from the probability of choosing a given k.

Most attempts at part (iii) were reasonably good, although many candidates got the binomial probabilities for the possible numbers of heads wrong.

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STEP MATHEMATICS 1

2018

Hints and Solutions

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Introductory Remarks

This document should be read in conjunction with the corresponding mark scheme in order to gain full benefit from it. Since the complete solutions appear elsewhere, much of this *Hints and Solutions* document will concentrate more on the "whys and wherefores" of the solution approach to each question and less on the technical details.

The solutions that follow, presented either in outline or in full, are by no means the only ones, not even necessarily the 'best' ones. They are simply intended to be the ones that, on the evidence of the marking process, appear to be the ones which arose most frequently from the ideas produced by the candidates and that worked for those who could force them through to a conclusion. If you "see" things in a different way, I hope you can still both follow and appreciate what is given here.

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Question 1

The first question on the first STEP is always intended to have a more familiar feel to it and this is one such question.

Equating the two expressions for y, for the two given entities to meet, then cancelling the obvious x, gives a simple quadratic: and you should know of a few ways to tackle it to find the coordinates (don't forget the y-coordinate) of both P and Q. The first "curve" is a line through the origin and the second is a cubic, also through O. The repeated factor in $y = x(b - x)^2$ indicates tangency at $x = b$ so the curve – a "positive" cubic, in the sense of the sign of the dominant (x^3) term – has the classical up-down-up shape, passes through O and touches the x-axis at b.

Next, "equation of tangent" should suggest that you to differentiate, substitute in the values of x and y at P, and build up the tangent's equation in the form required. Remember to *show clearly* how the answer is obtained; a lot of candidates lose marks by jumping to the given answer without justifying fully their arrival. A given answer always requires more detailed working; even if you are perfectly capable of doing all the working 'in your head', the working must be shown in writing.

The areas in the next part of the question can be found by integration but it is important to have a decently drawn diagram to point you in the right direction; especially as one of the regions in question is a triangle, which hardly requires the use of calculus (although, on this occasion, incorporating all into a single integral is just as straightforward).

Finally, now that you have algebraic expressions for S and T, you are asked to establish an inequality, $S > \frac{1}{3}T$. Inequalities can often be quite difficult to handle well. However, there is a sensible "trick" that often works well, and that is to prove an equivalent inequality involving zero: in this case, either $S - \frac{1}{3}T > 0$ or $3S - T > 0$. Establishing the sign of a single expression (such as $3S - T$) is almost always far more easily accomplished than having to compare a variable LHS with a variable RHS.

<u>Question 2</u>

In this question, the important thing is to be careful with the directions of the inequalities, and particularly the sign of anything that you intend to multiply or divide by (variable items are always tricky as they can be both positive and negative … at different "times", of course). Remember to say why you are doing what you are doing clearly; the biggest loss of marks in these sorts of questions invariably comes from those who write down statements, often correct ones, but the marker is unable to spot why they have been written down. Referring "invisibly" to a result a dozen lines up the page is not generally sufficient, and the markers are not required to hunt around and identify the reasons why you might have made such-and-such a conclusion and give you credit for their own understanding of the written work confronting them: spell it out clearly for them. If necessary, give each key result a circled number reference so you can cite them later on: see part **(iii) below** for an example.

In this question, to begin with, you are reminded of a result commonly referred to as the *Change-Of-Base-Formula*. Establishing it is easy provided you realise that the statements $x = \log_b c$ and $b^x = c$ are equivalent. Taking logs to base a for this second statement then gives the opening result.

In **(i)**, all that is needed is to take logs to base 10 with both the given (useable) statement $\pi^2 < 10$ and then with the LHS of the given inequality (using the *COBF* noted above). This gives (essentially) just the two lines of working: $\pi^2 < 10 \Rightarrow \log_{10}\pi^2 < 1 \Rightarrow 2\log_{10}\pi < 1 \Rightarrow \log_{10}\pi < \frac{1}{2}$ ① $\Rightarrow \dfrac{1}{\log_{10}\pi} > 2$

and $\dfrac{1}{\log_2\pi} + \dfrac{1}{\log_5\pi} = \dfrac{\log_{10}2}{\log_{10}\pi} + \dfrac{\log_{10}5}{\log_{10}\pi}$ by the COBF $= \dfrac{\log_{10}2 + \log_{10}5}{\log_{10}\pi} = \dfrac{\log_{10}10}{\log_{10}\pi} = \dfrac{1}{\log_{10}\pi} > 2$.

Part **(ii)** is a little more protracted, and this time the COBF needs to use logs to base e (i.e. ln's).

$\log_2\dfrac{\pi}{e} > \frac{1}{5} \Rightarrow \log_2\pi - \log_2 e > \frac{1}{5} \Rightarrow \dfrac{\ln\pi}{\ln 2} - \dfrac{\ln e}{\ln 2} > \frac{1}{5}$ by the COBF $\Rightarrow \ln\pi > 1 + \frac{1}{5}\ln 2$

and $e^2 < 8 \Rightarrow \ln e^2 < \ln 2^3 \Rightarrow 2\ln e < 3\ln 2 \Rightarrow \ln 2 > \frac{2}{3}$

so that, putting the two together, $\ln\pi > 1 + \frac{1}{5}\ln 2 > 1 + \frac{1}{5}.\frac{2}{3} = \frac{17}{15}$, as required.

Notice the way in which it is easiest to keep the direction of the inequalities consistent. This is a technique we shall continue to adopt in **(iii)**. Taking logs to base 10 (suggested by the presence of $\log_{10}2$), we have

$20 < e^3 \Rightarrow \log_{10}20 < 3\log_{10}e \Rightarrow \log_{10}10 + \log_{10}2 < 3\log_{10}e \Rightarrow 1 + \log_{10}2 < 3\log_{10}e$ ②

and $\pi^2 < 10 \Rightarrow \log_{10}\pi < \frac{1}{2}$ from ① above

and $\frac{3}{10} < \log_{10}2$ ③

It follows that $\frac{13}{10} < 1 + \log_{10}2 < 3\log_{10}e$ using ③ and ② so that $\frac{13}{30} < \log_{10}e = \dfrac{\ln e}{\ln 10} = \dfrac{1}{\ln 10} \Rightarrow \ln 10 < \frac{30}{13}$

and $\log_{10}\pi < \frac{1}{2}$ (①) $\Rightarrow \dfrac{\ln\pi}{\ln 10} < \frac{1}{2} \Rightarrow \ln\pi < \frac{1}{2}\ln 10 < \frac{1}{2}.\frac{30}{13} = \frac{15}{13}$.

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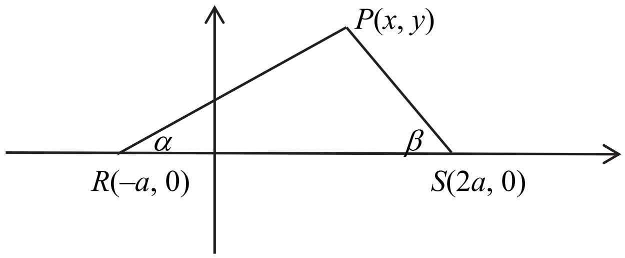
Question 3

This question involves a collection of fairly simple ideas, especially in part **(i)**, but leaves the candidate to explore the difference between "⇒" and "⇐" lines of reasoning with part **(ii)**. Again, at the outset, a good diagram can help in getting started:



Noting first that $\tan\alpha = \dfrac{y}{x+a}$ and $\tan\beta = \dfrac{y}{2a-x}$, if $\beta = 2\alpha$ then $\tan 2\alpha = \dfrac{y}{2a-x} = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{2\,\frac{y}{x+a}}{1-\frac{y^2}{(x+a)^2}}$.

Since $y \neq 0$, this all tidies up to the required $y^2 = 3\left(x^2 - a^2\right)$.

In **(ii)**, starting from $y^2 = 3\left(x^2 - a^2\right)$ leads backwards to $\tan\beta = \tan 2\alpha$. The difference now is that this gives the more general result, when "un-trigging" (think about your quadrants work, or the symmetries and periodicities of the tan function), that $\beta = 2\alpha + n\pi$. All that is needed now is to consider which integer values of n give a viable triangle setting for α and β (one of which must be the obvious $\beta = 2\alpha$, of course.)

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Question 4

Most of the difficulty inherent in this question is of a technical nature. It should be clear that you must differentiate and show that $(1 - \ln x)$ is a factor of both $f(x)$ *and* $f'(x)$, but finding the correct derivative will usually involve application of the product, quotient *and* chain rules.

In **(i)**, to begin with, the limits are an irrelevance, since the integrand is the same in both portions of the function F. Most of the time, in standard A-level papers, you are given the required substitution; not so here, although the only really obvious simplifying choice of $u = \ln t$ turns out to 'do the trick' perfectly well. The other thing that then turns out to be different from A-level is that, here, it is really important to integrate t^{-1} as $\ln|t|$; however, there is a big push to consider this when you realise that, in the two explicitly stated regions of $0 < x < 1$ and $x > 1$, the ln function is first negative and then positive. Since all powers of $\ln t$ are even, it turns out that

$$F(x) = \ln|\ln x| - (\ln x)^2 + \tfrac{1}{4}(\ln x)^4 + \tfrac{3}{4}$$

in each of the two regions. Indeed, the even powers, and the modulus, ensure that $F = F^{-1}$.

The sketch of the curve is interesting but all the clues are there already. There is a vertical asymptote of $x = 1$ (since $\ln(\ln 1) = \ln 0$ is inadmissible; thereafter, the final result of part **(i)** actually tells you that the curve exhibits a sort of reflection symmetry between the two regions. The final key observation is that each region has a point of inflexion as F crosses the x-axis.

Question 5

In many ways, as with a lot of STEP questions, the real difficulty here lies in having to put together, without a great deal (if any) of notification, a number of different mathematical ideas from almost anything you may have learnt over the years, but to do so in a suitably sophisticated way.

For a starter, you are asked for the *most general* quartic that leaves a remainder of 1 upon division by … . The required polynomial is clearly $k(x-1)(x-2)(x-3)(x-4)+1$ (the most common error being to think the leading coefficient is always 1) and it is helpful to note that k cannot be zero. This leads to the suggested $P(x) = k(x-1)(x-2)(x-3) \ldots (x-N)+1$, $k \neq 0$, in **(ii)** and substituting $x = N+1$ leads to the required conclusion that $P(N+1) \neq 1$ either by brute force or by a 'contradiction' argument that notes that, if this *were* the case, then we would have a polynomial of degree N with $(N+1)$ distinct roots.

For the very last part of **(ii)**, notice that you are only required to find one such r and, after a bit algebra, it becomes clear that $r = 2$ does the job; verifying that it does so is rather easy.

The same sorts of ideas then come into play in part **(iii)**, where we have a quartic polynomial equation, $S(x)-1=0$, with four distinct integer zeroes a, b, c and d. Setting $x = e$ then gives

$$(e-a)(e-b)(e-c)(e-d) = 17$$

and, although 17 *does* have four distinct integer factors (±1, ±17), the two 17s cannot both be used. In **(b)** a similar consideration of $S(0)$ leads to $abcd = 16$ so that only the numbers ±1, ±2, ±4, ±8, ±16 can be used. There are several possible approaches (three of which appear in the published Mark Scheme) but the most important things needed to be clear and systematic in your presentation.

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Question 6

This does *look* horrible (potentially, at least) and, in fact, it can indeed be solved in a very lengthy way if one is not careful. The opening result is a simple application of the well-known "Addition Formulae" for $\cos(A \pm B)$. Expanding the LHS of the given result so that each term on the LHS is of the form $2\sin P \sin Q$ leads to a "collapsing" (or "telescoping") series where almost all of the terms appear once negatively and once positively, leaving only the first and last ones standing.

This result is then used in **(i)**, once you have correctly split the area into n equal rectangular strips of equal bases but heights given by the height of each strip's midpoint: choosing the value of θ suitably and collapsing the series gives the required outcome, $A_n = \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{2n}$, which I have written in this form for later use.

The *Trapezium Rule* formula gives a similar result once you have factored in (to both sides, of course) the $\sin \frac{\pi}{2n}$, although a satisfactorily simplified answer requires the use of a final step involving $\cos(\pi - \theta) = -\cos\theta$ in order to end up with $B_n = \frac{\pi}{n} \cot \frac{\pi}{2n}$.

The two results of part **(iii)** now turn out to involve little more than bits of (unprompted) trig. identity work on the expressions for A_n and B_n in the above forms.

Question 7

To begin with, this is relatively straightforward: setting $x = \dfrac{pz+q}{z+1}$ in $x^3 - 3pqx + pq(p + q) = 0$ gives an expression which can be factorised into $(p - q)^2(pz^3 + q) = 0$ and, since $p \neq q$, it follows that $a = p$ and $b = q$.

In **(ii)**, we require $pq = c$ and $p + q = \dfrac{d}{c}$, and we can then consider p and q as the roots of the quadratic equation $y^2 - \dfrac{d}{c}y + c = 0$, which has two, real distinct roots provided its discriminant is positive.

Part **(iii)** is clearly going to be of the required form (but it is still important to demonstrate that this is so). It should be obvious, with a moment's thought, that there are going to appear to be two outcomes, but also that they will turn out to give the same result, since p and q are interchangeable. In the simpler version, we get $p = -1$, $q = 2$, which leads to $2 - z^3 = 0$ and $z = \sqrt[3]{2}$.

In **(iv)**, it is routine to find a linear factor (by using the factor theorem) and $x = p$ can be seen to give a zero of the given cubic expression. Proceeding as usual, one obtains $(x - p)^2(x + 2p) = 0$ and $x = p$ or $-2p$. These, in turn, yield the required roots of $x^3 - 3cx + d = 0$ in the case when $d^2 = 4c^3$.

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<u>Question 8</u>

What is especially pleasing about this question (from a teaching point of view) is that it principally deals with the simple details of calculus – both differentiation and integration – unencumbered by the specifics of individual functions (even though the "s" and "c" do look as if they have *something* to do with sine and cosine). In this sense, there is something fundamentally "pure" about the application of the various rules that are being effected.

In **(i)**, one must simply appreciate that a function differentiating to zero is constant and, in which case, whatever value one substitutes in, that constant will be the output. The required results in **(ii)** follow from the use of the product and quotient rules and the identity obtained in **(i)**.

The first result of **(iii)** follows directly from the given statement $c'(x) = -s(x)^2$, requiring only the most basic grasp of integration as anti-differentiation. For the second result, candidates need to identify how to split what appears to be the single term, needing to be integrated, into two useable "parts" and, following a bit of algebra, recognising a "reverse chain rule" integration (which, of course, can always be done with a suitable substitution).

In a similar vein, part **(iv)** tests your capacity to think in terms of the key elements of the substitution –integration process; while **(v)** then requires you to deploy a mixture of all of the above skills.

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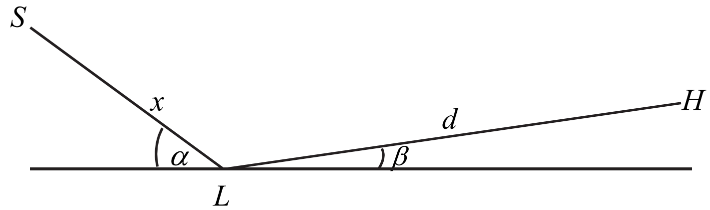
Question 9

A simple diagram can be remarkably effective in enabling you to put your thoughts in order, even if it consists of little more than this:



Now, in order to reach H from S, the GPE lost from S to L must exceed that gained from L to H (so that there is a non-negative amount of energy left for a non-negative kinetic energy). Thus

$$mg\,x\sin\alpha \geq mg\,d\sin\beta \quad \text{and} \quad x\sin\alpha \geq d\sin\beta .$$

The rest of the background work for the question can be done using the so-called "constant acceleration formulae",

$$s = ut + \tfrac{1}{2}at^2 \text{ ❶}, \quad v = u + at \text{ ❷}, \quad v^2 = u^2 + 2as \text{ ❸} \quad \text{and} \quad s = \tfrac{1}{2}(u+v)t \text{ ❹},$$

with suitably chosen values of the variables.

Using ❸ $v^2 = 2gx\sin\alpha \Rightarrow v = \sqrt{2gx\sin\alpha}$, and ❹ then gives $t_1 = \dfrac{2x}{v} = \sqrt{\dfrac{2x}{g\sin\alpha}}$.

Next, ❶ gives $d = vt_2 - \frac{1}{2}g\sin\beta\;t_2^{\,2}$, which is a quadratic equation in t_2 (and it is the first, smaller, root which is required). The bulk of the working for this main part of the question is then taken up in re-forming $T = t_1 + t_2$ into the given answer,

$$\left(\frac{g\sin\alpha}{2}\right)^{\frac{1}{2}} T = (1+k)\sqrt{x} - \sqrt{k^2x - kd}.$$

Once this has been obtained, we are required to do the usual sorts of calculus:

$$\frac{\mathrm{d}}{\mathrm{d}x}(RHS) = \frac{1+k}{2\sqrt{x}} - \frac{k^2}{2\sqrt{k^2x-kd}} = 0 \text{ when } \frac{(1+k)^2}{x} = \frac{k^4}{k^2x-kd}$$

$$\Rightarrow \left(1+2k+k^2\right)\left(kx-d\right) = k^3x \Rightarrow \left(k+2k^2+k^3-k^3\right)x = (1+k)^2 d$$

and $x = \dfrac{(1+k)^2}{k(1+2k)}d$.

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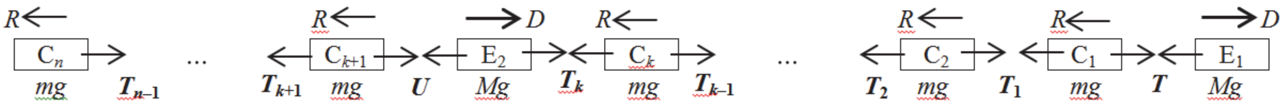
Question 10

A lot of mechanics questions are best approached with a helpful, carefully marked, diagram:



The rest of the question involves the application of N2L (*Newton's Second Law*) to various bits of this system, or the whole thing. The trick is to let what is asked for guide your thoughts to the best "bit" to choose to apply it to.

To begin with, apply N2L to the whole train: $2D - nR = (2M + nm)a$
and then to the first engine (E_1): $D - T = Ma$. Eliminating D gives the required result for **(i)**.

The next result may call for a bit of investigation, but each application of N2L takes time, so it is best to look at a representative carriage between E_1 and E_2 first, and then at one after E_2 (if necessary). The first of these gives $T_{r-1} - T_r = R + ma$, for $1 \le r \le k$, which is positive, so the tensions are decreasing. Then, the same principle will also apply to the carriages after E_2.

Next, use N2L for the final section of the train: $U - (n - k)R = (n - k)ma$.
Thereafter, we examine the quantity $T - U$, eliminate the D in the numerator of **(i)**'s expression for T, and end up with

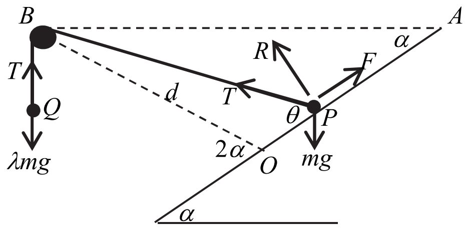
$$T - U = (k - \tfrac{1}{2}n)(R + ma)$$

which gives the required result in **(ii)** that $T > U$ provided $k > \frac{1}{2}n$.

Having seen that all the tensions in the couplings between the components in the front part of the train are positive, **if** the suggested outcome does occur, it must be in one of the carriages in the back part of the train. Try N2L for E_2: $D + T_k - U = Ma \Rightarrow T_k = U + Ma - D = U + Ma - (T + Ma) = U - T < 0$ from the result of **(ii)**.

Question 11

If the diagrams for the other mechanics questions were of questionable value, in questions such as this they are an absolute necessity:



In **(i)**, we can reason as follows: as $P \to O$, $\theta \to 2\alpha$ and as $P \to A$, $\theta \to \alpha$ so $\alpha \le \theta \le 2\alpha$.

In **(ii)**, the standard process of *resolution of forces* comes into play. These give

Res. || plane for P: $mg\sin\alpha + T\cos\theta = F$ ❶

Res. $\perp^r$. plane for P: $R + T\sin\theta = mg\cos\alpha$ ❷

Res. ↑ for Q: $T = \lambda mg$ ❸

Friction Law (eqlm.): $F \le \mu R$ ❹

and it follows that $R = mg\cos\alpha - T\sin\theta = mg(\cos\alpha - \lambda\sin\theta)$ by ❷ and ❸.

Since $R \ge 0$ (for P in contact with plane) for all values of θ, $\cos\alpha \ge \lambda\sin\theta$ as $\theta \to 2\alpha$; that is,

$$\cos\alpha > \lambda 2\sin\alpha\cos\alpha \Rightarrow 1 > 2\lambda\sin\alpha\,.$$

For **(iii)**, using $F \le \mu R$ (for now) with ❶ and ❸, $mg(\sin\alpha + \lambda\cos\theta) \le \tan\beta\,.mg(\cos\alpha - \lambda\sin\theta)$

$$\Rightarrow \quad \sin\alpha + \lambda\cos\theta \le \frac{\sin\beta}{\cos\beta}(\cos\alpha - \lambda\sin\theta)$$

Solving for λ then leads to $\lambda \le \dfrac{\sin\beta\cos\alpha - \sin\alpha\cos\beta}{\cos\theta\cos\beta + \sin\beta\sin\theta} = \dfrac{\sin(\beta-\alpha)}{\cos(\beta-\theta)}$, as required.

The very final section requires a bit of careful reasoning to arrive at the "corresponding result for $\alpha \le \beta \le 2\alpha$" of $\lambda \le \sin(\beta - \alpha)$.

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Question 12

The answer to **(i)** is clearly $\frac{1}{3}(p_1 + p_2 + p_3)$, which turns out to be the p referred to in the rest of the question.

In **(ii)**, and **(iii)** for that matter, there is no harm in setting out the table of outcomes and associated probabilities in tabular form

x	p(x)
2	p^2
1	$2p(1-p)$
0	$(1-p)^2$

and the individual probs. are relatively easy to work out. Then the standard results E(N_1) = $\sum x.\mathrm{p}(x)$ and Var(N_1) = $\sum x^2.\mathrm{p}(x) - (\mathrm{E}(N_1))^2$ give p and $2p(1-p)$ respectively, as required.

If it helps to put your thoughts in order, a tree diagram can help here – remember just to fill in the branches that are relevant – and the probabilities for the various numbers of heads that can arise are

x	p(x)
2	$\frac{1}{3}(p_1p_2 + p_2p_3 + p_3p_1)$
1	$\frac{1}{3}(p_1(1-p_2) + p_1(1-p_3) + p_2(1-p_1) + p_2(1-p_3) + p_3(1-p_1) + p_3(1-p_2))$
0	$\frac{1}{3}((1-p_1)(1-p_2) + (1-p_2)(1-p_3) + (1-p_3)(1-p_1))$

The application of the same standard results then give

$$\mathrm{E}(N_2) = 2p \text{ and } \mathrm{Var}(N_2) = 2p - 4p^2 + \tfrac{2}{3}(p_1p_2 + p_2p_3 + p_3p_1).$$

For **(iv)**, we show that Var(N_1) – Var(N_2) ≥ 0. This boils down to

$$2p^2 - \tfrac{2}{3}(p_1p_2 + p_2p_3 + p_3p_1) \geq 0$$

i.e. $\frac{2}{9}(p_1 + p_2 + p_3)^2 - \frac{2}{3}(p_1p_2 + p_2p_3 + p_3p_1) \geq 0$ or $2(p_1 + p_2 + p_3)^2 - 6(p_1p_2 + p_2p_3 + p_3p_1) \geq 0$ which, upon squaring, gives $2p_1^2 + 2p_2^2 + 2p_3^2 - 2(p_1p_2 + p_2p_3 + p_3p_1) \geq 0$. Now this expression is simply $(p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2$ and, being the sum of three squares, is guaranteed to be non-negative. Indeed, it is also immediately clear that equality only occurs when all three squared terms are zero; that is $p_1 = p_2 = p_3$.

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Question 13

For Candidate A, if $k \leq 2$, she can score a maximum of 4 marks so cannot pass. If $k = 3$, the only way to pass is by getting them all correct, with probability $\frac{1}{n^3}$. For $k = 4$, she can score 5 marks with 3 correct answers and 8 marks with 4 correct answers, giving probability ${}^4C_3 \frac{1}{n^3}\left(1 - \frac{1}{n}\right) + \frac{1}{n^4} = \frac{4n-3}{n^4}$. Then, finally, if $k = 5$, she can score 7 marks with 4 correct answers and 10 marks with 5 correct answers, so the probability of passing is ${}^5C_4 \frac{1}{n^4}\left(1 - \frac{1}{n}\right) + \frac{1}{n^5} = \frac{5n-4}{n^5}$. It now remains to demonstrate that $P_4 - P_3 > 0$ and that $P_4 - P_5 > 0$ to justify that $k = 4$ is best.

For Candidate B's strategy, we have a conditional probability:

$$\mathrm{P}(k = 4 \mid \text{pass}) = \frac{\mathrm{P}(k = 4 \,\&\, \text{pass})}{\mathrm{P}(\text{pass})} = \frac{\frac{1}{6} \times \frac{4n-3}{n^4}}{\left(\frac{1}{6} \times \frac{1}{n^3}\right) + \left(\frac{1}{6} \times \frac{4n-3}{n^4}\right) + \left(\frac{1}{6} \times \frac{5n-4}{n^5}\right)},$$

where the denominator consists of the terms $\mathrm{P}(k = 3 \,\&\, \text{pass})$, $\mathrm{P}(k = 4 \,\&\, \text{pass})$ and $\mathrm{P}(k = 5 \,\&\, \text{pass})$ respectively.

In the case of Candidate C, the probability of passing is just

$$\mathrm{P}(3H) \times \mathrm{P}(\text{pass} \mid 3H) + \mathrm{P}(4H) \times \mathrm{P}(\text{pass} \mid 4H) + \mathrm{P}(5H) \times \mathrm{P}(\text{pass} \mid 5H)$$
$$= {}^5C_3 \frac{n^3}{(n+1)^5} \times \frac{1}{n^3} + {}^5C_4 \frac{n^4}{(n+1)^5} \times \frac{4n-3}{n^4} + {}^5C_5 \frac{n^5}{(n+1)^5} \times \frac{5n-4}{n^5} = \frac{25n-9}{(n+1)^5}.$$

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STEP MATHEMATICS 1

2018

Mark Scheme

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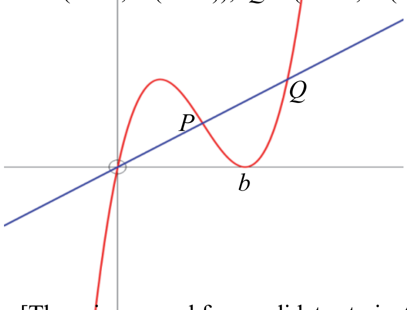
Q1	$a^2x = x(b-x)^2$	**M1 equating the two equations (with/without the factor o**	
	$\Rightarrow\ a = b - x$ or $-a = b - x$	**M1 for solving method, this way or via a quadratic equation … which should be $x^2 - (b^2 - a^2)$**	
	$x = b - a$ or $b + a$	**A1 both**	
	$P = (b - a, a^2(b - a)),\ Q = (b + a, a^2(b + a))$	**A1 both y-coordinates**	
		**B1 for a fully correct graph; N.B. $(b, 0)$ need not be noted**	**5**

[There is no need for candidates to justify that this is the correct arrangement: a second, more interesting, sketch arises when $0 < b < a$ but the question does not require it.]

$y = x^3 - 2bx^2 + b^2x \Rightarrow \dfrac{dy}{dx} = 3x^2 - 4bx + b^2$	**M1 for differentiating a cubic**	
or $\dfrac{dy}{dx} = (b-x)^2 - 2x(b-x)$	**using the *Product Rule* of differentiation on $y = x(b-x)^2$**	
$= 3(b^2 - 2ab + a^2) - 4b(b-a) + b^2$	**M1 for substituting $x = b - a$**	
$= 3a^2 - 2ab$ or $a(3a - 2b)$ at P	**A1 (AG) for correct gradient in any form**	
Eqn. of tgt. at P is $y - a^2(b-a) = a(3a-2b)(x - [b-a])$	**M1 method for tgt. eqn. via $y - y_c = m(x - x_c)$ or $y = mx + c$ with P's coords. substd.**	
$y = a(3a-2b)x + a^2(b-a) - (3a^2 - 2ab)(b-a)$		
$y = a(3a-2b)x - (b-a)[4a^2 - 2ab]$		
$y = a(3a-2b)x + 2a(b-a)^2$	**A1 (AG) legitimately obtained & written in this form**	**5**

$S = \displaystyle\int_0^{b-a}\left(x^3 - 2bx^2 + b^2x\right)dx - \tfrac{1}{2}a^2(b-a)^2$	**M1 method for finding area by ∫n. – Δ area**	
$= \left[\dfrac{1}{4}x^4 - \dfrac{2}{3}bx^3 + \dfrac{1}{2}b^2x^2\right]_0^{b-a} - \tfrac{1}{2}a^2(b-a)^2$	**B1 for correct ∫n. of a 3 (or 4) term cubic (even if Δ omitted)**	
$= \tfrac{1}{4}(b-a)^4 - \tfrac{2}{3}b(b-a)^3 + \tfrac{1}{2}b^2(b-a)^2 - \tfrac{1}{2}a^2(b-a)^2$	**M1 for substn. of correct limits in any integrated terms**	
$= \tfrac{1}{12}(b-a)^2\{3(b-a)^2 - 8b(b-a) + 6(b^2 - a^2)\}$	**M1 for correctly factoring out at least two linear terms**	
$= \tfrac{1}{12}(b-a)^3(3b - 3a - 8b + 6b + 6a)$	**(must have a difference of two areas or equivalent)**	
$= \tfrac{1}{12}(b-a)^3(b + 3a)$	**A1 (AG) legitimately obtained**	**5**

Area $\Delta OPR = \tfrac{1}{2}$ (y-coord. of R)×(x-coord. of P)	**M1 correct method for required area**	
$T = \tfrac{1}{2}.2a(b-a)^2.(b-a) = a(b-a)^3$	**A1 correct, factorised form for T seen at some stage**	
$\dfrac{S}{T} = \tfrac{1}{12}.\dfrac{b+3a}{a}$ or $S - \tfrac{1}{3}T = \ldots$ or $3S - T = \ldots$	**M1 for genuine attempt to consider any of these algebraically**	
	A1 (AG) correct result legitimately obtained	
$\dfrac{b+3a}{a} > \dfrac{a+3a}{a}\ \because\ b > a$	**E1 for proper justification of result**	
or $3S - T = \tfrac{1}{4}(b-a)^4 > 0\ \because\ b \neq a$	**(E0 for unexplained 'backwards' logic)**	**5**

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Q2	$c = b^x$	**B1**	
		M1 Taking logs to base a and rearranging	
(i)	$\log_{10} \pi^2 < 1$	**M1 Taking logs to base 10 of given inequality** **RHS might still be in terms of a log**	
		A1 Simplifying to an expression involving only one log **(might be awarded later)**	
	$\dfrac{1}{\log_2 \pi} + \dfrac{1}{\log_5 \pi} = \dfrac{\log_{10} 2}{\log_{10} \pi} + \dfrac{\log_{10} 5}{\log_{10} \pi}$	**M1 Writing both denominators in the same base** **(might not be base 10)**	
	$= \dfrac{1}{\log_{10} \pi}$	**M1 A1 Simplifying to an expression involving only one log**	
	Linking to given inequality to complete the proof that LHS > 2 **AG**	**E1 Penalise answers which assume the result here**	**6**
(ii)	$\ln \pi > 1 + \frac{1}{5} \ln 2$	**M1 Using the change-of-base-formula to turn into "ln"**	
		A1 Producing a correctly simplified version **(may be given implicitly later)**	
	$\ln 2 > \frac{2}{3}$	**M1 A1 Taking natural logs of given inequality**	
	Combining both facts to get $\ln \pi > \frac{17}{15}$ **AG**	**E1 Penalise answers which assume the result here**	**5**
(iii)	$\ln \pi < 1 + \frac{1}{2} \ln 10$	**M1 Taking a log of given inequality**	
	$= \dfrac{\log_{10} 10}{2 \log_{10} \mathrm{e}}$	**M1 Converting to base 10 using the change-of-base-formula**	
	$\log_{10} \mathrm{e} > \frac{1}{3} \log_{10} 20$	**M1 Taking log to base 10 of given inequality**	
	$= \frac{1}{3}(1 + \log_{10} 2)$	**M1 Linking to $\log_{10} 2$**	
	$> \frac{13}{30}$	**A1 Correct use of given result**	
	Putting it all together to get $\ln \pi < \frac{15}{13}$ **AG**	**E2 Penalise if inequality directions misused**	**7**

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Q3 (i)	$\tan\alpha = \frac{y}{x+a}$ and $\tan\beta = \frac{y}{2a-x}$.	**B1 B1**
	If $\beta = 2\alpha$ then	
	$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$	**M1 for using formula**
	$= \frac{\frac{2y}{x+a}}{1-\frac{y^2}{(x+a)^2}}$	**A1 correct unsimplified**
	$= \frac{y}{2a-x}$	**M1 equating with $\tan\beta$**
	i.e. $\frac{y}{2a-x} = \frac{2y(x+a)}{(x+a)^2-y^2}$	**A1 simplified equation**
	$y((x+a)^2 - y^2) = 2y(x+a)(2a-x)$	**M1 for getting rid of fractions**
	$(x+a)^2 - y^2 = 2(x+a)(2a-x)$ since $y > 0$	**E1 for justifying this step (this could happen earlier)**
	$x^2 + a^2 + 2ax - y^2 = 4a^2 - 2x^2 + 2ax$ so	
	$3x^2 - 3a^2 = y^2$	**A1 (AG)**
	Alt.1:	
	$y = \mathrm{PR}\sin\alpha = \mathrm{PS}\sin 2\alpha$ so $\mathrm{PR} = 2\mathrm{PS}\cos\alpha$	**M1 M1 for useful expression for $\cos\alpha$**
	$x + a = PR\cos\alpha = 2PS\cos^2\alpha$	
	$2a - x = \mathrm{PS}\cos 2\alpha = 2\mathrm{PS}\cos^2\alpha - \mathrm{PS}$	
	so $3x - 3a = 2\mathrm{PS}(1-\cos^2\alpha) = 2\mathrm{PS}\sin^2\alpha$	**M1 A1 A1 for expressing x^2 and y^2 in terms of a and a length**
	so $3(x^2-a^2) = 4\mathrm{PS}^2\sin^2\alpha\cos^2\alpha = y^2$.	**M1 A1 for expression for $3(x^2-a^2)$**
		M1 A1 (AG) for checking equality
	Alt.2:	
	Let angle bisector of S meet PR at T.	
	PST and PRS are similar	**B1**
	so PT/PS = PS/PR	**M1 A1**
	$PT = PR\frac{x-a/2}{x+a}$, and so	**M1 A1**
	$PR^2\left(x-\frac{a}{2}\right) = PS^2(x+a)$	**M1**
	Pythagoras gives	**M1 A1 unsimplified cubic**
	$((x-2a)^2+y^2)(x+a) = ((x+a)^2+y^2)\left(x-\frac{a}{2}\right)$	
	Simplifying: $\frac{3a}{2}y^2 = \frac{9a}{2}(x^2-a^2)$	**A1 (AG)**
	For methods (not involving similar triangles) which reach a higher-order polynomial in a, x and y, give **M1 M1 A1**. *As progress towards this, give* **M1 M1** *for Sine Rule + Pythagoras.*	
	Alt.3:	
	$y = \tan\alpha\,.\,(x+a)$ and $y = \tan\beta\,.\,(2a-x)$	**B1 B1**
	$\tan\alpha\,.\,(x+a) = \frac{2\tan\alpha}{1-\tan^2\alpha}(2a-x)$	**M1 for double tangent A1**
	$\tan\alpha \neq 0$ so $x + a = \frac{2a-x}{1-\tan^2\alpha}$	**E1**
	giving $x = \frac{3+\tan^2\alpha}{3-\tan^2\alpha}a$	**M1 A1 writing x in terms of a and $\tan\alpha$**
	and $y = \cdots$	**M1 A1 for expression for y and checking $y^2 = 3(x^2-a^2)$** **9**
(ii)	If $3(x^2-a^2) = y^2$ then	
	$(x+a)^2 - y^2 = 2(x+a)(2a-x)$	**M1 rearranging into something useful**
	$x \neq 2a, -a$ (the latter because $y > 0$)	**E1 justifying this**
	meaning both sides non-zero so	
	$\frac{y}{2a-x} = \frac{2y(x+a)}{(x+a)^2-y^2}$	**M1 dividing through**
	$= \frac{\frac{2y}{x+a}}{1-\frac{y^2}{(x+a)^2}}$	**M1 A1 for something in terms of $\tan\alpha$**
	So $\tan\beta = \tan 2\alpha$	**A1**

Some candidates might just say "everything in (i) is reversible", without checking. I suggest such a claim would get the three M marks above but not the A or E marks. If for some reason a candidate does this part but not (i), they should also get the two B1 marks and the first M1 from part (i) for using these facts here.

Other methods exist which give instead $\cos 2\alpha = \pm\cos\beta$.

Alt.:

$\tan(\beta-\alpha)=\dfrac{\frac{y}{2a-x}-\frac{y}{x+a}}{1+\frac{y^2}{(x+a)(2a-x)}}$	**M1**	
$=\dfrac{y(2x-a)}{(x+a)(2a-x)+y^2}$	**M1 for single fraction**	
$=\dfrac{y(2x-a)}{(x+a)(2a-x)+3x^2-3a^2}$	**M1 substituting y**	
$=\dfrac{y(2x-a)}{(x+a)(2x-a)}$	**A1**	
Since $x\neq a/2$ (as otherwise $y^2<0$)	**E1 (be generous if there is an attempt to justify)**	
we get $\tan(\beta-\alpha)=\tan\alpha$	**A1**	**6**
This means $\beta=2\alpha+k\pi$ for some integer k.	**B1**	
$0<\alpha<\pi$ so $y>0$ and $0<\beta<\pi$ **OR** $y>0$ and $x<2a$ so $0<\beta<\pi/2$	**B1 for bounding β (the bound you get depends on whether you use the information given in this part or given earlier)**	
so $-\pi<2\alpha-\beta<2\pi$	**M1 for using this to bound k**	
so $k=0,-1$	**A1 only two values of k – don't worry about a sign error**	
giving $\beta=2\alpha$ or $\beta=2\alpha-\pi$.	**A1 cao (don't need to check both are possible)**	**5**
Alt. part (ii) (all 11 marks):		
Construct the point $S'=(2x-2a,0)$, making PSS$'$ isosceles.	**M2***	
Now $\text{PS}^2=y^2+(2a-x)^2$		
$=3(x^2-a^2)+(2a-x)^2$		
$=(2x-a)^2=\text{RS}'^2$	**M2* A2***	
Thus we have $\text{PS}=\text{PS}'=\text{RS}'$		
If S$'$ lies between R and S, this gives $\text{RPS}'=\alpha$ and $\text{PS}'\text{R}=\pi-2\alpha$ so $\beta=2\alpha$.	**M1 A1**	
If R lies between S$'$ and S, this gives	**B1 for considering both cases**	
$\text{PRS}'=(\pi-\beta)/2$ so $\beta=2\alpha-\pi$.	**M1 A1**	

Candidates who attempt this are likely to do all the calculations separately for the two cases. If so, give 1 mark out of each 2 above for each part where the corresponding working appears.*

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Q4

$$\mathrm{f}'(x) = \frac{x\ln x.2\left(1-(\ln x)^2\right).-2\ln x.\frac{1}{x} - \left(1-(\ln x)^2\right)^2.\left(x.\frac{1}{x}+\ln x\right)}{\left(x\ln x\right)^2}$$

M1 Use of product or quotient rule (or alt. substn.)
A1 1st term (numerator) correct
A1 2nd term (numerator) & denominator correct

Showing both f(x) and $\mathrm{f}'(x) = 0$ when $(\ln x)^2 = 1$ — **E1** — **4**

(i) $u = \ln t$ — **M1 Any sensible substitution**

$I = \displaystyle\int \frac{(1-u^2)^2}{u}\,\mathrm{d}u$ — **M1 A1 Full substitution used; correct** $= \displaystyle\int\left(\frac{1}{u} - 2u + u^3\right)\mathrm{d}u$

$= \ln|u| - u^2 + \frac{1}{4}u^4 \quad (+\,c)$ — **A1 Penalise absence of modulus signs here (but allow for next 2 marks)**

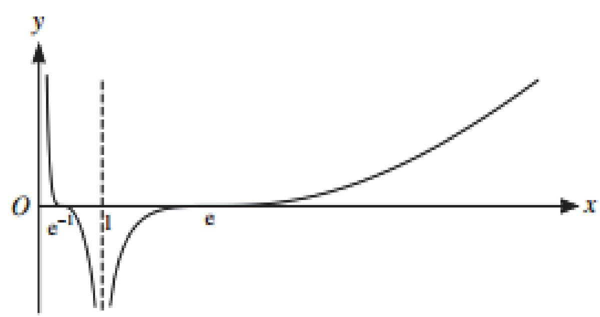
$= \ln|\ln x| - (\ln x)^2 + \frac{1}{4}(\ln x)^4 + \frac{3}{4}$ for $0 < x < 1$ — **A1**

$= \ln|\ln x| - (\ln x)^2 + \frac{1}{4}(\ln x)^4 + \frac{3}{4}$ for $x > 1$ — **A1** — **6**

$\mathrm{F}(x^{-1}) = \ln|-\ln x| - (-\ln x)^2 + \frac{1}{4}(-\ln x)^4 + \frac{3}{4}$
$= \mathrm{F}(x)$

M1 For using $\ln(x^{-1}) = -\ln x$
E1 For candidates who notice that F(x) takes the same functional form, this will be quite easy. Otherwise, two cases are required. — **2**

(ii)



G1 Asymptote $x = 0$
G1 Asymptote $x = 1$
G1 Negative gradient for $0 < x < 1$
G1 Positive gradient for $x > 1$
G1 Stationary points at $x = \mathrm{e}^{-1}$ and $x = \mathrm{e}$
G1 Points of inflexion at $x = \mathrm{e}^{-1}$ and $x = \mathrm{e}$
G1 Zeroes at $x = \mathrm{e}^{-1}$ and $x = \mathrm{e}$
G1 Generally correct shape — **8**

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Q5	(i)	$k(x-1)(x-2)(x-3)(x-4)+1$	**B1 no need to state that $k \neq 0$**	
	(ii)	$\mathrm{P}(x) = k(x-1)(x-2)(x-3)\ \ldots\ (x-N)+1$ $\mathrm{P}(N+1) = k(N)(N-1)(N-2)\ \ldots\ (1)+1$ $= k(N\,!)+1 = 1$ iff $k=0$ **Alt.** $\mathrm{P}(x) = 1$ is a polynomial of degree N so has N roots, 1 to N inclusive; but if $P(N+1) = 1$ also then it has $N+1$ … a contradiction $\mathrm{P}(N+1) = 2$ iff $k = \dfrac{1}{N!}$ $\mathrm{P}(N+r) = \dfrac{1}{N!}(N+r-1)\,(N+r-2)\,(N+r-3)\ \ldots\ (r)+1$ $= \dfrac{(N+r-1)!}{N!(r-1)!}+1$ or $\dbinom{N+r-1}{N}+1$ Let $m = N+r$ (so that $m > N$) Require $\mathrm{P}(m) = \dbinom{m-1}{N}+1 = m$ or $\dbinom{m-1}{N} = m-1$ $\dbinom{m-1}{N} = \dfrac{(m-1)(m-2)(m-3)\ldots(m-N)}{N(N-1)\ldots\times 2} = m-1$ $\Rightarrow\ m = N+2$ i.e. $r = 2$ <u>Notes:</u> Question only requires candidates to *find* a suitable r so noting $r = 2$ (**M1**) and checking that it works (**M1 A1**) can score all of these final 3 marks	**B1** **M1 P($N+1$) for an N^{th}-degree polynomial** **A1** **A1 (no k, no mark)** **B1 any form** **M1 or equivalent statement** **M1 for general approach** **A1**	**8**
	(iii)	$\mathrm{S}(x) = (x-a)(x-b)(x-c)(x-d)+2001$	**B1 stated (a, b, c, d distinct integers)**	**1**
	(a)	$\mathrm{S}(e) = (e-a)(e-b)(e-c)(e-d)+2001 = 2018$ $\Rightarrow\ (e-a)(e-b)(e-c)(e-d) = 17$ $\Rightarrow$ 17 has (at least) 4 distinct integer factors However, 17 has only four factors; ±1, ±17 and, as both 17s cannot be used, no such integer e exists	**M1** **A1** **M1 looking at factorisations of 17** **A1 must be fully explained**	**4**
	(b)	$\mathrm{S}(x) = (x-a)(x-b)(x-c)(x-d)+2001$ (integers $a < b < c < d$) $\mathrm{S}(0) = abcd + 2001 = 2017\ \Rightarrow\ abcd = 16$ **B1** and we require 16 to be written as the product of four distinct integers a, b, c, d with $a < b < c < d$ Thus $a, b, c, d \in \{\pm1, \pm2, \pm4, \pm8, \pm16\}$; allow $\{\pm1, \pm2, \pm4, \pm8\}$ **M1** **I** If $a = -16$, then $b, c, d \in \{\pm1\}$ and this cannot be done distinctly **II** If $a = -8$, then $b, c, d \in \{\pm1, \pm2\}$ with exactly one of them $-_{\text{ve}}$ $\Rightarrow\ (a, b, c, d) = $ **(–8, –1, 1, 2)** **III** If $a = -4$, then $b, c, d \in \{\pm1, \pm2, \pm4\}$ with exactly one of them $-_{\text{ve}}$ $\Rightarrow\ (a, b, c, d) = $ **(–4, –2, 1, 2)** or **(–4, –1, 1, 4)** **IV** If $a = -2$, then $b, c, d \in \{\pm1, \pm2, \pm4, \pm8\}$ with exactly one of them $-_{\text{ve}}$ $\Rightarrow\ (a, b, c, d) = $ **(–2, –1, 2, 4)** or **(–2, –1, 1, 8)** **V** $a \neq -1$ since then $abcd < 0$ and if $a > 0$ then $abcd \geq 64$ There are thus 5 ways in which a, b, c, d can be chosen s.t. $\mathrm{S}(0) = 2017$ **M1 for a (partially) systematic case analysis** **A1 for any three correct solutions** **A1 for all five and no extras** **E1 for correct justification no solutions in cases I, V** <u>Important note:</u> Candidates need to identify clearly the *number* of cases (so the actual solutions are not required) and may still gain the marks despite numerical errors if the method for finding them is clearly explained. However, I very much doubt this will happen.		

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Alt. 1 The cases could be argued by sign first and then value, as follows.

a, b, c, d cannot be all $+_{ve}$ or all $-_{ve}$ since then $abcd \geq 64$

so we must have two $+_{ve}$ and two $-_{ve}$.

Note that $|a| \neq 16$ since all three others must then have $|..| = 1$.

So the options are:

I $(a, b) = (-8, -4)$ impossible since $abcd$ already too big

II $(a, b) = (-8, -2)$ impossible since then both c, d must equal 1

III $(a, b) = (-8, -1) \Rightarrow (c, d) = (1, 2)$

IV $(a, b) = (-4, -2) \Rightarrow (c, d) = (1, 2)$

V $(a, b) = (-4, -1) \Rightarrow (c, d) = (1, 4)$

VI $(a, b) = (-2, -1) \Rightarrow (c, d) = (1, 8)$ or $(2, 4)$

and there are thus 5 ways in which a, b, c, d can be chosen s.t. $S(0) = 2017$

M1 for a (partially) systematic case analysis

A1 for any three correct solutions

A1 for all five and no extras

E1 for correct justification no solutions in cases I, II

Alt. 2 Instead, one might reason thus:

As a product of four factors, in magnitude order,

$16 = 1.1.1.16$ or $1.1.2.8$ or $1.1.4.4$ or $1.2.2.4$ or $2.2.2.2$

We reject the first and last of these since we can have at most two of equal magnitude (two $+_{ve}$ and two $-_{ve}$). This leaves us with

I $1.1.2.8$ gives $(a, b, c, d) = \{-1, 1, -2, 8\}$ or $\{-1, 1, 2, -8\}$

i.e. $(a, b, c, d) = (-2, -1, 1, 8)$ or $(-8, -1, 1, 2)$

II $1.1.4.4$ gives $(a, b, c, d) = \{-1, 1, -4, 4\}$

i.e. $(a, b, c, d) = (-4, -1, 1, 4)$

III $1.2.2.4$ gives $(a, b, c, d) = \{-2, 2, 1, -4\}$ or $\{-2, 2, -1, 4\}$

i.e. $(a, b, c, d) = (-4, -2, 1, 2)$ or $(-2, -1, 2, 4)$

M1 for a (partially) systematic case analysis

A1 for any three correct solutions

A1 for all five and no extras

E1 for initial justification which 4-term factorisations of 16 work

6

Q6 $2\sin\theta\left(\sin\theta+\sin 3\theta+\sin 5\theta+\ldots+\sin(2n-1)\theta\right)$

$\equiv 2\sin\theta\sin\theta+2\sin\theta\sin 3\theta+2\sin\theta\sin 5\theta+\ldots+2\sin\theta\sin(2n-1)\theta$

$\equiv \left(\cos 0-\cos 2\theta\right)+\left(\cos 2\theta-\cos 4\theta\right)+\left(\cos 4\theta-\cos 6\theta\right)+\ldots$ **M1 complete method using given identity**

$\ldots+\left(\cos(2n-2)\theta-\cos 2n\theta\right)$

$\equiv 1-\cos 2n\theta$ since all intermediate terms cancel **A1 (AG) legitimately obtained** **2**

(i) The midpoint of the k^{th} strip is at $x=\dfrac{(k-\frac{1}{2})\pi}{n}$ or $\dfrac{(2k-1)\pi}{2n}$ **B1**

Ht. of strip is $\sin\dfrac{(2k-1)\pi}{2n}$ and its area is $\dfrac{\pi}{n}\sin\dfrac{(2k-1)\pi}{2n}$ **B1**

A_n = sum of all strips

$=\dfrac{\pi}{n}\left(\sin\theta+\sin 3\theta+\sin 5\theta+\ldots+\sin(2n-1)\theta\right),\ \theta=\dfrac{\pi}{2n}$ **M1**

$=\dfrac{\pi}{n}\left(\dfrac{1-\cos 2n\theta}{2\sin\theta}\right)$ **M1 for using the initial result**

$=\dfrac{\pi}{n}\left(\dfrac{1-\cos\pi}{2\sin\theta}\right)=\dfrac{\pi}{n}\left(\dfrac{2}{2\sin\left(\dfrac{\pi}{2n}\right)}\right)$

$\Rightarrow A_n\sin\dfrac{\pi}{2n}=\dfrac{\pi}{n}$ **A1 (AG) fully established** **5**

(ii) $B_n=\frac{1}{2}\left(\dfrac{\pi}{n}\right)\left\{\sin 0+2\left(\sin\left(\dfrac{\pi}{n}\right)+\sin\left(\dfrac{2\pi}{n}\right)+\sin\left(\dfrac{3\pi}{n}\right)+\ldots+\sin\left(\dfrac{(n-1)\pi}{n}\right)\right)+\sin\pi\right\}$

M1 use of Trapezium Rule formula in context

$=\left(\dfrac{\pi}{n}\right)\left(\sin\theta+\sin 2\theta+\sin 3\theta+\ldots+\sin(n-1)\theta\right),\ \theta=\dfrac{\pi}{n}$

$B_n\sin\dfrac{\pi}{2n}=\left(\dfrac{\pi}{n}\right)\sin\frac{1}{2}\theta\left(\sin\theta+\sin 2\theta+\sin 3\theta+\ldots+\sin(n-1)\theta\right)$

$=\left(\dfrac{\pi}{2n}\right)\left(2\sin\frac{1}{2}\theta\sin\theta+2\sin\frac{1}{2}\theta\sin 2\theta+2\sin\frac{1}{2}\theta\sin 3\theta+\ldots+2\sin\frac{1}{2}\theta\sin(n-1)\theta\right)$

M1 use of initial result

$=\left(\dfrac{\pi}{2n}\right)\left\{\left(\cos\frac{1}{2}\theta-\cos\frac{3}{2}\theta\right)+\left(\cos\frac{3}{2}\theta-\cos\frac{5}{2}\theta\right)+\ldots+\left(\cos(n-\frac{3}{2})\theta-\cos(n-\frac{1}{2})\theta\right)\right\}$

$=\left(\dfrac{\pi}{2n}\right)\left\{\cos\left(\dfrac{\pi}{2n}\right)-\cos(n-\frac{1}{2})\left(\dfrac{\pi}{n}\right)\right\}$ **A1 all intermediate terms cancelled**

Now, $\cos(n-\frac{1}{2})\left(\dfrac{\pi}{n}\right)=\cos\left(\pi-\dfrac{\pi}{2n}\right)=-\cos\left(\dfrac{\pi}{2n}\right)$ **M1 dealing with the final term in { }**

so $B_n\sin\dfrac{\pi}{2n}=\left(\dfrac{\pi}{2n}\right)\left\{2\cos\left(\dfrac{\pi}{2n}\right)\right\}=\dfrac{\pi}{n}\cos\left(\dfrac{\pi}{2n}\right)$ **A1** **5**

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(iii) $A_n + B_n$

$$= \frac{\pi}{n\sin\left(\frac{\pi}{2n}\right)} + \frac{\pi\cos\left(\frac{\pi}{2n}\right)}{n\sin\left(\frac{\pi}{2n}\right)} = \frac{\pi}{n\sin\left(\frac{\pi}{2n}\right)}\left(1+\cos\left(\frac{\pi}{2n}\right)\right)$$ **M1**

$$= \frac{\pi}{n\sin\left(\frac{\pi}{2n}\right)}\left(1+2\cos^2\left(\frac{\pi}{4n}\right)-1\right)$$ **M1 use of double-angle formula**

$$= \frac{2\pi\cos^2\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}$$ **A1 or equivalent later tidying up**

$$B_{2n} = \frac{\pi\cos\left(\frac{\pi}{4n}\right)}{2n\sin\left(\frac{\pi}{4n}\right)} = \frac{\pi\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)}{n.2\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)} = \frac{\pi\cos^2\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}$$ **M1 B_n result with $n \to 2n$**

$= \frac{1}{2}(A_n + B_n)$ as required **A1 (AG) fully established**

Alt. $A_n = \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{2n}$ and $B_n = \frac{\pi}{n}\cot\frac{\pi}{2n}$, so the result now boils down to trig. identity work:

$A_n + B_n = \frac{\pi}{n}\left(\frac{1+\cos\theta}{\sin\theta}\right)$, $\theta = \frac{\pi}{2n}$ **M1**

$= \frac{1+2c^2-1}{2sc} = \frac{c}{s}$ where $c = \cos\frac{1}{2}\theta$ and $s = \sin\frac{1}{2}\theta$ **M1 A1 using half-angle results**

$= 2\frac{\pi}{2n}\cot\frac{\pi}{4n} = 2B_{2n}$ **A1 B1** **5**

$$A_n B_{2n} = \frac{\pi}{n\sin\left(\frac{\pi}{2n}\right)} \times \frac{\pi\cos^2\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}$$ using the work above

$$= \left[\frac{\pi\cos\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}\right]^2$$ **B1 sensible expression for LHS**

$$A_{2n} = \frac{\pi}{2n\sin\left(\frac{\pi}{4n}\right)} = \frac{\pi\cos\left(\frac{\pi}{4n}\right)}{n.2\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)} = \frac{\pi\cos\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}$$ **M1 attempt at RHS**

Since all terms are positive, we can take positive square roots to get the required result,

$\sqrt{A_n B_{2n}} = A_{2n}$ **A1 (AG) fully established**

Alt. Done via trig. identity work: $A_n B_{2n} = \frac{\pi}{n} \cdot \frac{1}{2sc} \cdot \frac{\pi}{2n} \cdot \frac{c}{s} = \frac{\pi^2}{4n^2 s^2} = \left(\frac{\pi}{2n} \cdot \frac{1}{s}\right)^2 = (A_{2n})^2$ etc.

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Q7 (i) If $x = \frac{pz+q}{z+1}$ and $x^3 - 3pqx + pq(p+q) = 0$ then

$\left(\frac{pz+q}{z+1}\right)^3 - 3pq\left(\frac{pz+q}{z+1}\right) + pq(p+q) = 0$ so — **M1 substitution**

$(pz+q)^3 - 3pq(pz+q)(z+1)^2$
$\quad + pq(p+q)(z+1)^3 = 0$ — **M1 multiplying by $(z+1)^3$**

i.e. $(p^3 - 3p^2q + p^2q + pq^2)z^3$ — **M1 for expanding & collecting like terms**

$+(3p^2q - 3pq^2 - 6p^2q + 3p^2q + 3pq^2)z^2$ — **A1 (AG) for checking middle terms vanish**

$+(3pq^2 - 3p^2q - 6pq^2 + 3p^2q + 3pq^2)z$

$+(q^3 - 3pq^2 + p^2q + pq^2) = 0$ — **A1 for correct first/last terms (do not need to divide by $(p-q)^2$ to get this mark, but it will help later!)**

i.e. $(p-q)^2(pz^3+q) = 0$ and $p \neq q$ so

$pz^3 + q = 0$

Alt.: *write $z = \frac{x-q}{p-x}$ (**M1**) and substitute in $az^3 + b = 0$ (**M1**). Expand and collect terms (**M1**) and check $\frac{b}{a} = \frac{q}{p}$ for no quadratic term (**A1**). Initial cubic legitimately obtained (**A1**).* **5**

(ii) We need $pq = c$ and $pq(p+q) = d$ so $p + q = \frac{d}{c}$ — **M1 conditions on p, q**

These are roots of a quadratic $y^2 - \frac{d}{c}y + c = 0$ — **M1 A1**

This has distinct real roots iff $\left(\frac{d}{c}\right)^2 - 4c > 0$ — **M1 for evaluating discriminant**

Since $c^2 > 0$, iff $d^2 > 4c^3$. — **A1 (AG)**

Candidates who evaluate $d^2 - 4c^3$ in terms of p and q and show the inequality holds just get first M1 (this is the converse). By writing $p = c/q$ or vice versa it is possible to get a quadratic for one of them, but unless they justify that p, q distinct when the discriminant is positive, don't give the final A1. Another alternative is to calculate $(p-q)^2 = \frac{d^2}{c^2} + 4c$ and use this to deduce values for p and q (this is equivalent to the normal solution). **5**

(iii) We need $p + q = 1$ and $pq = -2$ so $p = 2, q = -1$ — **M1 A1 (using quadratic or by inspection)**

So this reduces to $2z^3 - 1 = 0$ and $z = 2^{-1/3}$ — **M1 A1 ft for value of z**

and $x = \frac{2z-1}{z+1} = \frac{2^{2/3}-1}{2^{-1/3}+1}$ — **M1 A1 ft calculating x**

OR … so $p = -1, q = 2$

So this reduces to $2 - z^3 = 0$ and $z = 2^{1/3}$ — **(only one of these needed)**

and $x = \frac{2-z}{z+1} = \frac{2-2^{1/3}}{2^{1/3}+1}$ (equivalent to above) **6**

(iv) $x = p$ is a root; — **M1 spotting (any) root**

factoring gives $(x-p)(x^2 + p - 2p^2) = 0$ — **A1**

and $(x-p)(x-p)(x+2p) = 0$ so $x = p, -2p$ — **A1**

Thus the equation reduces to the above with

$p = \frac{d}{2c}$ so has roots $x = \frac{d}{2c}, \frac{-d}{c}$. — **A1 ft**

*Equivalent values of p, such as $\sqrt[3]{d/2}$ are fine here, but **NOT** $p = \sqrt{c}$ as this isn't necessarily the correct sign.* **4**

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Q8

(i) $\frac{d}{dx}\left(s(x)^3 + c(x)^3\right) = 3s(x)^2 s'(x) + 3c(x)^2 c'(x)$ **using the *Chain Rule* of differentiation M1**

$= 3s(x)^2.c(x)^2 + 3c(x)^2.-s(x)^2 = 0 \Rightarrow s(x)^3 + c(x)^3 = \text{constant}$ **A1**

Since $s(0)^3 + c(0)^3 = 0^3 + 1^3 = 1,\ s(x)^3 + c(x)^3 = 1$ for all x **A1** **3**

(ii) $\frac{d}{dx}\left(s(x)c(x)\right) = s(x)c'(x) + s'(x)c(x)$ **using the *Product Rule* of differentiation**

$= s(x)^2.-s(x)^2 + c(x)^2.c(x)$ **with derivatives substd. M1**

$= c(x)^3 - s(x)^3 = c(x)^3 - \left[1 - c(x)^3\right]$ using **(i)**'s result **A1**

$= 2c(x)^3 - 1$ **must show that given answer is obtained from (i)** **2**

$\frac{d}{dx}\left(\frac{s(x)}{c(x)}\right) = \frac{c(x)s'(x) - s(x)c'(x)}{c(x)^2}$ **using the *Quotient Rule* of differentiation**

$= \frac{c(x)c(x)^2 - s(x).-s(x)^2}{c(x)^2}$ **with derivatives substd.** **M1**

$= \frac{c(x)^3 + s(x)^3}{c(x)^2} = \frac{1}{c(x)^2}$ using **(i)**'s result **m.s.t.g.a.i.o.f.(i)** **A1** **2**

(iii) $\int s(x)^2\,dx = -c(x) + K$ **ignore missing *K*'s throughout** **B1** **1**

$\int s(x)^5\,dx = \int s(x)^3 s(x)^2\,dx$ **correct splitting and …**

$= \int \left[1 - c(x)^3\right] s(x)^2\,dx$ using **(i)**'s result **… use of (i) or ∫n. by parts*** **M1**

$= \int s(x)^2\,dx - \int c(x)^3.-c'(x)\,dx$ **or use of parts twice** **M1**

$= -c(x) + \frac{1}{4}c(x)^4 + K$ **using "reverse *Chain Rule*" integration** **A1**

NB* $I = \int s^5\,dx = \int s^3 s^2\,dx = s^3.-c + \int 3s^2c^3\,dx = -c\,s^3 + 3\int s^2(1 - s^3)\,dx$

$= -c\,s^3 + 3\int s^2\,dx - 3I \Rightarrow 4I = -c\,s^3 - 3c \Rightarrow I = -\frac{3}{4}c(x) - \frac{1}{4}c(x)s(x)^3 + K$ **3**

(iv) $u = s(x) \Rightarrow du = s'(x)\,dx = c(x)^2\,dx$ & $1 - u^3 = 1 - s(x)^3 = c(x)^3$ **full substn. prepn. B1**

$\int \frac{1}{\left(1 - u^3\right)^{\frac{2}{3}}}\,du = \int \frac{1}{c(x)^2}.c(x)^2\,dx = \int 1\,dx = x + K = s^{-1}(u) + K$ **M1 A1** **3**

(v) $\int \frac{1}{\left(1 - u^3\right)^{\frac{4}{3}}}\,du = \int \frac{1}{c(x)^4}.c(x)^2\,dx$ **full substn.** **M1**

$= \int \frac{1}{c(x)^2}\,dx = \frac{s(x)}{c(x)} + K$ using **(ii)**'s result **A1 in *s/c* (ft sign)** $= \frac{u}{\left(1 - u^3\right)^{\frac{1}{3}}} + K$ **A1 in *u*** **3**

$\int \left(1-u^3\right)^{\frac{1}{3}}\ \mathrm{d}u = \int c(x).c(x)^2\ \mathrm{d}x = \int c(x)^3\ \mathrm{d}x$	**full substn.**	**M1**	
$= \int \left(\frac{1}{2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[s(x)c(x)\right]\right) \mathrm{d}x$	using **(ii)**'s result	**M1**	
$= \frac{1}{2}x + \frac{1}{2}s(x)c(x) + K = \frac{1}{2}s^{-1}(u) + \frac{1}{2}u\left(1-u^3\right)^{\frac{1}{3}} + K$		**A1**	**3**

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Q9	$mg\,x\sin\alpha \ge mg\,d\sin\beta$	**M1 Attempt to use conservation of energy**	
	$\Rightarrow\ x\sin\alpha \ge d\sin\beta$	**A1 Correct**	

Acceleration down the slope is $g\sin\alpha$	**B1**	
$v^2 = 2g\,x\sin\alpha$	**M1A1 Use of appropriate kinematic formula; correct**	
$t_1 = \dfrac{x}{\frac{1}{2}\sqrt{2gx\sin\alpha}}$ or $\sqrt{\dfrac{2x}{g\sin\alpha}}$	**A1**	
Acceleration up the slope is $-g\sin\beta$	**B1 Clear use of correct sign convention required**	
$d = v\,t_2 - \frac{1}{2}g\sin\beta\ t_2^{\,2}$	**M1 A1 Use of appropriate kinematic formulae.** **NB: this and the previous M1A1 can also be gained from conservation of energy considerations**	
$t_2 = \dfrac{v \pm \sqrt{v^2 - 2dg\sin\beta}}{g\sin\beta}$	**M1 A1 Use of the quadratic formula; correct**	
Justifying taking the negative sign	**E1**	
	M1 Algebraic working towards correct form	
$\left(\dfrac{g\sin\alpha}{2}\right)^{\frac{1}{2}} T = (1+k)\sqrt{x} - \sqrt{k^2x - kd}$	**A1 Given Answer convincingly obtained**	**12**

$\dfrac{\mathrm{d}}{\mathrm{d}x}(RHS) = \dfrac{1+k}{2\sqrt{x}} - \dfrac{k^2}{2\sqrt{k^2x-kd}}$	**M1 A1 Finding the derivative; correct**	
$= 0$ when $\dfrac{(1+k)^2}{x} = \dfrac{k^4}{k^2x-kd}$	**M1 Setting derivative to zero and attempting to solve**	
$(1+k)^2\left(k^2x-kd\right) = k^4x$	**M1 Sensible separating and squaring**	
$x = \dfrac{(1+k)^2}{k(1+2k)}d$	**M1 A1 Isolating x; correct**	**6**

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Q10

	Working	Marks	
(i)	$2D - nR = (2M + nm)a \Rightarrow a = \dfrac{2D - nR}{2M + nm}$	**M1 A1 Use of N2L for the train; *a* correct**	
	$D - T = Ma$	**B1 N2L for the front engine**	
	$T = \dfrac{D(2M + nm) - M(2D - nR)}{2M + nm}$	**M1 Combining results**	
	$= \dfrac{n(mD + MR)}{2M + nm}$	**A1 Getting Given Answer legitimately**	**5**
(ii)	For the r^{th} carriage, with $1 \le r \le k$	**M1 Considering a general carriage between the two engines**	
	$T_{r-1} - T_r - R = ma$	**A1**	
	$T_{r-1} - T_r = R + ma > 0 \Rightarrow$ tensions decreasing	**E1**	
	Noting the same applies after the 2nd engine	**E1**	
	If U is the tension of the connection to 2nd engine …	**M1 Considering the tension just after 2nd engine (can be done in several ways)**	
	$U - (n - k)R = (n - k)ma$	**A1**	
	Then $T - U = \dfrac{n(mD + MR)}{2M + nm} - (n - k)(R + ma)$	**M2 Finding an expression for *T* – *U***	
	From first line, $2D - nR = (2M + nm)a$ $\Rightarrow 2mD - mnR = (2M + nm)ma$ $\Rightarrow 2mD + 2MR = (ma + R)(2M + nm)$ Substituting in: $T - U = \frac{1}{2}n(R + ma) - (n - k)(R + ma)$	**M1 Reasonable strategy for dealing with the algebra; such as eliminating *D*. Don't reward those going round in circles or reverse logic.**	
	$= (k - \frac{1}{2}n)(R + ma)$	**A1 Getting to a correct factorised expression**	
	So $T > U$ if $k > \frac{1}{2}n$	**E1 Given Answer suitably justified**	**11**
(iii)	For the 2nd engine, $T_k + D - U = Ma$	**M1 A1 N2L for the 2nd engine**	
	$\Rightarrow T_k = U + Ma - D = U - T$	**M1 Eliminating *D* using N2L on 1st engine ($T = D - Ma$)**	
	From above, $T > U$ if $k > \frac{1}{2}n$ so $T_k < 0$	**E1 Correct justification using (ii)'s result**	**4**

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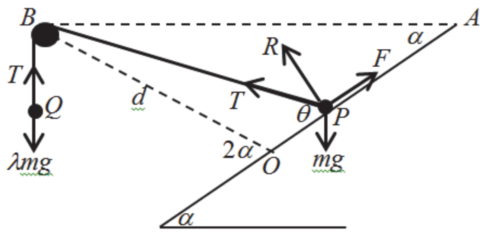
Q11 (i) $\angle BAO = \alpha$ (due to || lines) — **E1 Accept this just labelled on a diagram**

$\angle ABO = \alpha$ (due to Isos.Δ) so $\angle BOA = 180° - 2\alpha$ — **E1 Or accept Ext.$\angle$ of Δ**

So when P is at O, $\theta = 2\alpha$

and when P is at A, $\theta = \alpha$ and the result follows — **E1 (since qn. says *P* is between *O* and *A*)** **3**

(ii) Labelled diagram:



B1

$R + T\sin\theta = mg\cos\alpha$ — **M1 A1 Resolving perpr. to plane for *P***

$T = \lambda mg$ — **B1 Resolving vertically for freely-hanging mass**

$R > 0$ if P is in contact with the plane,

so $\cos\alpha \ge \lambda\sin\theta$ — **E1**

If this is true for all values then it holds at the largest possible θ … so $\cos\alpha \ge \lambda\sin 2\alpha$ — **E1 Condone no mention that $\alpha < 45°$**

i.e. $\cos\alpha \ge \lambda \, . \, 2\sin\alpha\cos\alpha \Rightarrow 1 \ge 2\lambda\sin\alpha$ — **M1 Use of trig. identity and "cancelling" to get Given Answer**
Condone oversight of checking for division by zero **7**

(iii) $mg\sin\alpha + T\cos\theta = F$ — **M1 A1 Resolving parallel to plane; correct**

$F = mg\tan\beta(\cos\alpha - \lambda\sin\theta)$ — **B1 Correct use of $F \le \mu R$ (condone =)**

$$\sin\alpha + \lambda\cos\theta = \frac{\sin\beta}{\cos\beta}(\cos\alpha - \lambda\sin\theta)$$

$$\Rightarrow \sin\alpha\cos\beta + \lambda\cos\theta\cos\beta = \sin\beta\cos\alpha - \lambda\sin\beta\sin\theta$$

$$\Rightarrow \lambda(\cos\theta\cos\beta + \sin\beta\sin\theta) = \sin\beta\cos\alpha - \sin\alpha\cos\beta$$

$$\Rightarrow \lambda = \frac{\sin\beta\cos\alpha - \sin\alpha\cos\beta}{\cos\theta\cos\beta + \sin\beta\sin\theta}$$ — **M1 Isolating λ**

$$= \frac{\sin(\beta - \alpha)}{\cos(\beta - \theta)}$$ — **M1 A1 Use of compound angle formula; answer correct**

Since $\theta < 2\alpha$ and $\beta \ge 2\alpha$ then $\beta - \theta > 0$ — **E1 Explaining how given condition is used**

The minimum of $\sec(\beta - \theta)$ is achieved when θ is a maximum; i.e. $\theta = 2\alpha$. For the system to be in equilibrium for all P between O and A then λ must be less than this — **E1 Explaining how considering $\theta = 2\alpha$ leads to the necessary condition**

If $\alpha \le \beta \le 2\alpha$ then the minimum of $\sec(\beta - \theta)$ is achieved when $\theta = \beta$ — **E1**

at which point the condition becomes $\lambda \le \sin(\beta - \alpha)$ — **B1** **10**

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Q12 (i) $P(H) = \frac{1}{3}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3$ **B1**

NB follow through marks below are for candidates who have a probability of $p_1 + p_2 + p_3$ above, and work with this as the probability of an individual head, but check that they actually have $(1 - p_1 - p_2 - p_3)$ for the probability of a tail, not $(3 - p_1 - p_2 - p_3)$; the latter is a wrong method and gets nothing.

(ii)

value	prob
2	p^2
1	$2p(1-p)$
0	$(1-p)^2$

M1 A1 ft (need P(2) and either P(1) or P(0), but if they give all three, require all correct)

gives $\mathrm{E}(N_1) = 2p^2 + 2p(1-p) + 0 = 2p$ **A1 cao**

$\mathrm{Var}(N_1) = \mathrm{E}(N_1^2) - \mathrm{E}(N_1)^2$ **M1 for this or other plausible method**

and $\mathrm{E}(N_1^2) = 4p^2 + 2p(1-p) + 0 = 2p^2 + 2p$ **A1 ft for this or other intermediate calculation for Var**

so $\mathrm{Var}(N_1) = 2p(1-p)$ **A1 (AG)**

Alt. *approach is to argue that these are twice the corresponding values for one toss (**M1 A1**) where $E(X) = E(X^2) = p$ (**M1 A1**), then getting the two values (**A1 A1 AG**),*
Or *just to say this is* $\mathrm{Bin}(2,p)$ *(**M2**) and quote the mean (**A2**) and variance (**A2**) for that (which is in formula book: to get marks for this candidates MUST explicitly write* $\mathrm{Bin}(2,p)$ *).* **6**

(iii)

value	prob
2	$(p_1p_2 + p_2p_3 + p_3p_1)/3$
1	$[p_1(1-p_2) + p_2(1-p_1) + p_2(1-p_3) + p_3(1-p_2) + p_3(1-p_1) + p_1(1-p_3)]/3$
0	$[(1-p_1)(1-p_2) + (1-p_2)(1-p_3) + (1-p_3)(1-p_1)]/3$

M1 for working out probabilities by conditioning on the chosen coins
A1 for at least one prob correct
A1 for all correct (of at least two given)

gives $\mathrm{E}(N_2) = \cdots = 2p$ **A1 ft correct expression + A1 cao simplified**

$\mathrm{Var}(N_2) = \mathrm{E}(N_2^2) - \mathrm{E}(N_2)^2$ **M1 for this or other plausible method**

and $\mathrm{E}(N_2^2) = 2p + 2(p_1p_2 + p_2p_3 + p_3p_1)/3$ **A1 ft for this or other intermediate calculation for Var**

so $\mathrm{Var}(N_2) = 2p + \frac{2(p_1p_2+p_2p_3+p_3p_1)}{3} - 4p^2$ **A1 cao any equivalent expression**

Alt. *is to calculate probs for each pair of coins (**M1 A1 A1**) then $E(N)$ and $E(N^2)$ for each pair of coins (**A1 A1**), then average at this point to give $E(N_2)$ and $E(N_2^2)$ (**M1 A1**), then calculate variance (**A1**). For a correct evaluation of the expectation by conditioning on which coins are chosen, but no probabilities/variance, give* **M1 A1 A1**. **8**

(iv) Look at $\mathrm{Var}(N_1) - \mathrm{Var}(N_2) =$

$2(p_1^2 + p_2^2 + p_3^2 - p_1p_2 - p_2p_3 - p_3p_1)/9$ **B1 ft for suitable simplified expression**

$= ((p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2)/9$ **M1 for attempt to complete square A1 for partial completion**

≥ 0, with equality iff **A1 for full completion to get the inequality**

$p_1 - p_2 = p_2 - p_3 = p_3 - p_1 = 0$, i.e $p_1 = p_2 = p_3$ **E1 for justifying when equality occurs**

Anyone who uses the rearrangement inequality here is likely to get full marks – but check final E1!
An example of "partial completion" is writing as $p_1(p_1 - p_2) + p_2(p_2 - p_3) + p_3(p_3 - p_1)$ from which the result would follow by w.l.o.g.-ing $p_1 \geq p_2 \geq p_3$. **5**

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Q13 (i)	If $k \le 2$ she can get at most 4 marks, so P(pass)=0	**B1 for ruling out $k < 3$**	
	If $k = 3$ the only way to pass is 3 right answers, with probability $\frac{1}{n^3}$.	**B1 for needs all correct if k = 3** **(give this even if prob. wrong/missing)**	
	If $k = 4$, 3 or 4 correct will pass, and this has prob.	**M1 for 3/4 needed & attempt to get prob. (but see remark)**	
	$\frac{4(n-1)}{n^4} + \frac{1}{n^4} = \frac{4n-3}{n^4}$.	**A1* correct prob.**	
	If $k = 5$, 4 or 5 correct will pass, and this has prob.	**M1 for 4/5 needed & attempt to get prob. (but see remark)**	
	$\frac{5(n-1)}{n^5} + \frac{1}{n^5} = \frac{5n-4}{n^5}$.	**A1* correct prob.**	**6**
	$\frac{4n-3}{n^4} - \frac{1}{n^3} = \frac{3(n-1)}{n^4} > 0$ since $n > 1$.	**M1 comparing via difference/quotient A1 inequality justified**	
	$\frac{4n-3}{n^4} - \frac{5n-4}{n^5} = \frac{4n^2-8n-4}{n^5} = \frac{4(n-1)^2}{n^5} > 0$	**M1 A1 for correct difference, A1 for justifying inequality**	
	So $k = 4$ is best.		**5**

NB *It is possible to do part (i) without calculating any probabilities: if you would pass with three questions, answering another question cannot hurt you, and if you would fail after four questions, answering another question cannot help you. A candidate who explains this will get most of the marks above, but do not give the two marks* **A1*** *unless these probabilities appear later on – you do need to calculate these probabilities at some point.*

(ii)	$\mathrm{P}(k = 4 \mid \text{pass}) = \frac{\mathrm{P}(k=4 \cap \text{pass})}{\mathrm{P}(\text{pass})}$	**M1 for this in any form**	
	$= \frac{\frac{1}{6}\times\frac{4n-3}{n^4}}{\frac{1}{6}\times\frac{4n-3}{n^4}+\frac{1}{6}\times\frac{1}{n^3}+\frac{1}{6}\times\frac{5n-4}{n^5}}$	**M1 A1ft for substituting probs from before or re-calculating**	
	$= \frac{4n^2-3n}{5n^2+2n-4}$	**A1cao (simplified to a quotient of polys, not nec. lowest terms)**	
	If a candidate jumps straight to the second line, assume they know where it comes from. However, if they jump straight to the second line without the $\frac{1}{6}$s (and without justifying that they cancel), withhold the final **A1**.		**4**
(iii)	$\mathrm{P}(\text{pass}) = \mathrm{P}(3\text{ heads})\mathrm{P}(\text{pass} \mid 3\text{ heads}) + \mathrm{P}(4\text{ heads})\mathrm{P}(\text{pass} \mid 4\text{ heads}) + \mathrm{P}(5\text{ heads})\mathrm{P}(\text{pass} \mid 5\text{ heads})$	**M1 for conditioning on the number of heads**	
	$= 10\frac{n^3}{(n+1)^5}\times\frac{1}{n^3} + 5\frac{n^4}{(n+1)^5}\times\frac{4n-3}{n^4} + \frac{n^5}{(n+1)^5}\times\frac{5n-4}{n^5}$	**M1 A1 for calculating binomial probabilities**	
		M1 A1cao for substituting probs from before or re-calculating	**5**

If a candidate works throughout with a particular value of n (typically 2) they can get at most the following marks: B1 B1 M1 A0 M1 A0 (4/6) M1 A0 M0 A0 A0 (1/5) M1 M0 A0 A0 (1/4) M1 M1 A1 M1 A0 (4/5), total 10.

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